Reply to "Comment on 'Transition to turbulence in a shear flow' "

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The preceding Comment by Basombrio and Campo challenges the numerical calculations and the conclusions on the variations of lifetimes with parameters presented by Eckhardt and Mersmann [Phys. Rev. E **60**, 509 (1999)]. The authors claim that the singular variations are ruled out by standard mathematical results and that the numerical accuracy is insufficient to support our conclusions. We will show that the first claim is based on a misinterpretation of the theorems and that the second line of reasoning ignores the structural origin of the singularities.

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In Ref. [2] we presented a low-dimensional model for the dynamics of shear flows that become turbulent without a linear instability of the laminar profile. The turbulent dynamics was found to be transient. We studied it by picking initial conditions and determining the time it took to return to the laminar profile. The studies show a sensitive dependence on initial conditions and wild variations in lifetimes, which led us to suggest that this was due to an underlying fractal set of singularities. The Comment by Basombrio and Campo [1] challenges the numerical calculations and the conclusions on the variations of lifetimes with parameters presented in [2]. The authors claim that the singular variations are ruled out by standard mathematical results and that the numerical accuracy is insufficient to support our conclusions. We will show that the first claim is based on a misinterpretation of the theorems and that the second line of reasoning ignores the structural origin of the singularities.

(1) The authors claim that "nonfractality is a direct consequence of the *continuous dependence on initial values* of the original problem" (quoted from [1]). To support this they quote a mathematical theorem applicable for "any compact set in the extended phase space" [1]. Such results do not apply here, as the relevant extended phase space, consisting of the state variables of the system and the time coordinate, is not compact: The lifetime problem intrinsically asks for times going to infinity, and then the smooth variation is no longer given. Consider the differential equation $\dot{x}=\lambda x$ with the solution $x(t)=x_0 \exp(\lambda t)$. The equivalent of the lifetime measurements in [2] and other papers [3–6] is to take $|x_0|$ small and to ask for the time it takes to reach some large value L, i.e., $|x|>L$. This time is $T=(1/\lambda)\ln(L/x_0)$, and it passes through infinity when $x_0 \to 0$. The theorem quoted in the Comment requires a finite (extended) phase space, so it does not contain the complete trajectories from the initial value to the threshold if the initial value is too close to zero.

(2) The structural reason for the existence of the singularities in lifetimes as claimed in [2] is the existence of hyperbolic saddles in phase space. In [2] only stationary states that appear at higher Reynolds number were studied. The presence of periodic orbits at lower Reynolds numbers, as detected by Schmiegel [7], is mentioned in the Conclusions. A more complete analysis of a related model is given in [8].

If a hyperbolic structure exists in phase space, the motion along an unstable manifold is described by the simple differential equation given under the previous item. Therefore, trajectories very close to the stable manifold will have large lifetimes and those on the stable manifold will have infinite lifetimes.

If there is not just a single hyperbolic saddle but a Smale horseshoe type structure, then there will be infinitely many stable manifolds and a sampling of initial conditions crossing all these manifolds will have a dense set of singularities.

Finite time tracking of a trajectory cannot establish permanent trapping. But the wild fluctuations of lifetimes under parameter variation combined with the presence of hyperbolic objects make the existence of singularities in lifetimes very plausible. On the quantitative side, results of [9] show that the long lifetimes imply that the stable manifolds are almost spacefilling, so that the one-dimensional line of initial conditions will most likely cross it an infinite number of times.

(3) Following a trajectory for a long time in a chaotic system is plagued by the exponential sensitivity implied by a positive Lyapunov exponent. The numerical examples in the Comment show this very clearly. How errors are amplified depends on specifics of the system, and in some cases a numerical trajectory can remain close to a true one for unexpectedly large times [10].

In view of this it is important to realize that the key to the behavior of the system is not the numerical demonstration of infinite lifetimes (which is impossible), but the presence of the hyperbolic elements. Typically, these elements are either stationary [11,12] or simply periodic [13], and require calculations for finite times only. The relation between the exact objects and the numerical approximation is then controlled by the usual considerations in numerical analysis. For the finite differential equations studied in [2,4,8] Newton's method gives fixed points to within machine precision and periodic solutions to better than 10^{-10} with not too much effort.

(4) Finally, I would like to point out that these studies are the partial differential analog of the chaotic scattering investigations in ordinary differential equations, triggered by stud-

ies in chemical physics, astrophysics, and hydrodynamics [14–17], reviewed in [18–20]. There, simple models have been developed [21,22] that make the singular variations of lifetimes very transparent.

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